PERGAMON

International Journal of Solids and Structures 38 (2001) 6227–6240

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

Interface crack tip field in a kind of rubber materials

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Received 18 October 1999; in revised form 26 September 2000

Abstract

The interface crack tip field between two dissimilar rubber materials those obey the Knowles–Sternberg (J. Elast. 3 (1973) 67–107) elastic law is analysed. The whole field composed of two shrinking sectors and one expanding sector. Under tensile and shear mixed load the interface is always located in the expanding sector provided the hardening exponents of the materials are equal. The completely analytical solutions are obtained for both shrinking and expanding sectors. It is found that the important expanding sector was ignored by Herrmann's (J. Elast. 21 (1989) 227–269) solution so that the location of interface cannot be determined. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords: Rubber materials; Interface crack; Tip field

1. Introduction

In plastic materials, the crack tip behavior is very sensitive to the material character. In rubber materials, when large strain is considered, the crack tip fields possess similar feature for quite different elastic laws given by Knowles–Sternberg (1973) and Gao (1990, 1997), i.e. the stress state is uniaxial tension. The completely analytical solution to the crack tip field is given by Gao and Gao (1999) that is valid for all of the three elastic laws mentioned above. Besides, the common feature of crack tip field is explained by Gao and Gao (1999).

As for the interface crack, some problems must be treated individually for different materials. Analysed by Gao and Shi (1995) is the rubber material that obeys the elastic law of Gao (1990). Analysed by Herrmann (1992) is the material that obeys Knowles–Sternberg (1973) elastic law. Since the important expanding sector was ignored by Herrmann (1989, 1992), a very long discussion was caused but the condition to determine the location of interface was not given. Besides, some restrictions on material constants given by Hermann (1989, 1992) are unreasonable ($(A_L/A_u)^{n/(2n-1)} = C_u B_L / C_L B_u$). In the present paper, the interface crack problem for Knowles–Sternberg's elastic law will be analysed. A rapid transition zone in

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PII: S0020-7683(00)00375-9

between the expanding sector and shrinking sector is found, that can ensure the continuity of displacements and stress.

2. Basic equations

Consider a three dimensional elastic body. Before and after deformation, the position vectors of a material point are denoted by $\boldsymbol{P}$ and $\boldsymbol{Q}$ respectively. The Lagrangian coordinates is denoted by x^i $(i = 1, 2, 3)$. Two sets of local triads are defined

$$\boldsymbol{P}_i = \frac{\partial \boldsymbol{P}}{\partial x^i}, \qquad \boldsymbol{Q}_i = \frac{\partial \boldsymbol{Q}}{\partial x^i} \tag{1}$$

Then, two invariants that will be used can be introduced

$$I = (\boldsymbol{P}^i \cdot \boldsymbol{P}^j)(\boldsymbol{Q}_i \cdot \boldsymbol{Q}_j), \qquad J = \det|\boldsymbol{P}^i \cdot \boldsymbol{Q}_j| \tag{2}$$

in which $\boldsymbol{P}^i$ is the conjugate of $\boldsymbol{P}_i$, i.e. $\boldsymbol{P}^i \cdot \boldsymbol{P}_j = \delta^i_j$.

A strain energy per undeformed unit volume was proposed by Knowles–Sternberg (1973)

$$U = (AI + BJ + CIJ^{-2})^n \tag{3}$$

where A, B, C and n are material constants. Form U the Cauchy stress can be obtained

$$\boldsymbol{\sigma} = \frac{1}{J} \frac{\partial U}{\partial \boldsymbol{Q}_i} \otimes \boldsymbol{Q}_i \tag{4}$$

in which $\otimes$ is the dyadic symbol.

Noting that

$$\frac{\partial I}{\partial \boldsymbol{Q}_i} \otimes \boldsymbol{Q}_i = 2\boldsymbol{d}, \qquad \frac{\partial J}{\partial \boldsymbol{Q}_i} \otimes \boldsymbol{Q}_i = J\mathbf{1} \tag{5}$$

where

$$\begin{cases} \boldsymbol{d} = P^{ij}\boldsymbol{Q}_i \otimes \boldsymbol{Q}_j, \quad P^{ij} = \boldsymbol{P}^i \cdot \boldsymbol{P}^j \\ \mathbf{1} = \boldsymbol{P}^i \otimes \boldsymbol{P}_i = \boldsymbol{Q}^i \otimes \boldsymbol{Q}_i \end{cases} \tag{6}$$

$\boldsymbol{d}$ is the Cauchy strain tensor, $\boldsymbol{1}$ is unit tensor. Then Eqs. (3) and (4) give

$$\boldsymbol{\sigma} = \frac{2n}{J}(AI + BJ + CIJ^{-2})^{n-1}\left[(A + CJ^{-2})\boldsymbol{d} + J\left(\frac{B}{2} - IJ^{-3}\right)\mathbf{1}\right] \tag{7}$$

The equilibrium equation is

$$\boldsymbol{Q}^i \cdot \frac{\partial}{\partial x^i}\sigma = 0 \tag{8}$$

If the base area forces are introduced

$$\boldsymbol{T}^i = V_Q\, \boldsymbol{\sigma} \cdot \boldsymbol{Q}^i \tag{9}$$

$$\text{where} \quad V_Q = (\boldsymbol{Q}_1, \boldsymbol{Q}_2, \boldsymbol{Q}_3) \tag{10}$$

the brackets denote mixed product of the three vectors. Then Eq. (8) can be rewritten as

$$\frac{\partial \boldsymbol{T}^i}{\partial x^i} = 0 \tag{11}$$

3. Division of sectors

Consider a semi-infinite interface crack between two dissimilar rubber materials as shown in Fig. 1(a). Under the action of tensile and shear mixed load, it deformed as shown in Fig. 1(b). Two cylindrical Lagrangian coordinate are taken such that (R, Θ, Z) refers to undeformed state, while (r, θ, z) refers to deformed state, as shown in Fig. 1. The problem is treated as plane strain case, and z coordinate is along the crack front. Then $z \equiv Z$, so only the mapping from R, Θ to r, θ is considered.

Since the strain near the crack tip is tremendous large, the deformation cannot be described by an uniform mapping function. The whole crack tip field is divided into two shrinking sectors SH, SH′ and one expanding sector EX as shown in Fig. 1. Before loading the sectors SH and SH′ almost occupy the whole field but after loading they become very narrow. Oppositely, before loading the sector EX is very narrow at the vicinity of the interface but after loading it occupies almost the whole field. In sectors EX and SH (or SH′) the mapping function will take different forms.

Actually there is no strict boundary between different sectors. The division of sectors only due to their characters of deformations.

4. Expanding sector

The EX sector contains two different material domains for which every quantity will be denoted by a subscript number 1 or 2 in necessary case, but in general case the subscript will be omitted. In this sector the mapping function is assumed as

$$\begin{cases} r = R^{1+\beta}\rho(\xi), \quad \xi = \Theta R^{-\alpha} \\ \theta = \omega(\xi) \end{cases} \tag{12}$$

where α, β, are positive exponents to be determined. $|\Theta| < \Theta_0$, Θ_0 is a very small positive number. The physical meaning of Eq. (12) is that the angle Θ is expanded tremendously while R is shrinked tremendously. Let $\boldsymbol{e}_r, \boldsymbol{e}_\theta$ denote the unit vectors of r, θ system, i.e.

$$\boldsymbol{e}_r = \boldsymbol{Q}_r = \frac{\partial \boldsymbol{Q}}{\partial r}, \qquad \boldsymbol{e}_\theta = \frac{1}{r}\boldsymbol{Q}_\theta = \frac{1}{r}\frac{\partial \boldsymbol{Q}}{\partial \theta} \tag{13}$$

then according to Eqs. (1), (12) and (13) it follows

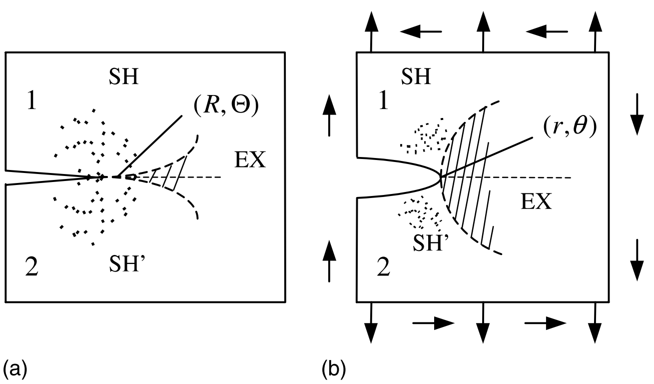

Fig. 1. Interface crack tip: (a) before loading, (b) after loading.

$$\begin{cases} \boldsymbol{Q}_R = \frac{\partial \boldsymbol{Q}}{\partial R} = R^{\beta}(1+\beta)\rho\boldsymbol{e}_r \\ \boldsymbol{Q}_\xi = \frac{\partial \boldsymbol{Q}}{\partial \xi} = R^{1+\beta}(\rho'\boldsymbol{e}_r + \rho\omega'\boldsymbol{e}_\theta) \end{cases} \tag{14}$$

Noting that

$$\boldsymbol{P}^R = \boldsymbol{e}_R = \boldsymbol{P}_R, \qquad \boldsymbol{P}^\xi = R^{-1-\alpha}\boldsymbol{e}_\Theta = R^{-2-2\alpha}\boldsymbol{P}_\xi \tag{15}$$

where $\boldsymbol{e}_R, \boldsymbol{e}_\Theta$ are the unit vectors in R, Θ system. Then from Eqs. (6), (14) and (15), it follows

$$\boldsymbol{d} = R^{2\beta-2\alpha}[\rho'^2\boldsymbol{e}_r \otimes \boldsymbol{e}_r + \rho^2\omega'^2\boldsymbol{e}_\theta \otimes \boldsymbol{e}_\theta + \rho\rho'\omega'(\boldsymbol{e}_r \otimes \boldsymbol{e}_\theta + \boldsymbol{e}_\theta \otimes \boldsymbol{e}_r)] + \mathrm{O}(R^{2\alpha}) \tag{16}$$

According to Eqs. (2), (14) and (15) it follows

$$\begin{cases} I = R^{2\beta-2\alpha}u(\xi) + \mathrm{O}(R^{2\alpha}) \\ J = R^{2\beta-2\alpha}v(\xi) \end{cases} \tag{17}$$

where

$$\begin{cases} u = \rho'^2 + \rho^2\omega'^2 \\ v = (1+\beta)\rho^2\omega' \end{cases} \tag{18}$$

Since $\alpha > 0$, evidently from Eq. (17) that $I \gg J$. Further assume that $\alpha > 2\beta$, then $J \gg 1$. There Eq. (7) is reduced to

$$\boldsymbol{\sigma} = \frac{2n}{J}(AI)^{n-1}\left[A\boldsymbol{d} + J\left(\frac{B}{2} - CIJ^{-3}\right)\mathbf{1}\right] \tag{19}$$

For the time being we assume that $I \sim J^3$ then from Eq. (17) it follows

$$\alpha = 4\beta \tag{20}$$

Using Eqs. (16)–(20), it follows

$$\begin{aligned} \boldsymbol{\sigma} = 2nA^nR^{-(2n-1)(\alpha-\beta)-\beta}u^{n-1}v^{-1}[&\rho'^2\boldsymbol{e}_r \otimes \boldsymbol{e}_r + \rho^2\omega'^2\boldsymbol{e}_\theta \otimes \boldsymbol{e}_\theta + \rho\rho'\omega'(\boldsymbol{e}_r \otimes \boldsymbol{e}_\theta + \boldsymbol{e}_\theta \otimes \boldsymbol{e}_r) \\ &+ R^\alpha vY(\boldsymbol{e}_r \otimes \boldsymbol{e}_r + \boldsymbol{e}_\theta \otimes \boldsymbol{e}_\theta)] \end{aligned} \tag{21}$$

in which

$$Y = \left(\frac{B}{2A} - \frac{C}{A}uv^{-3}\right) \tag{22}$$

In Eq. (21) the terms with R^α are reserved but the terms with $R^{2\alpha}$ are neglected. From Eqs. (14) and (10) it follows

$$\begin{cases} \boldsymbol{Q}^R = R^{-\beta}v^{-1}(\rho\omega'\boldsymbol{e}_r - \rho'\boldsymbol{e}_\theta) \\ \boldsymbol{Q}^\xi = R^{-\beta-1}v^{-1}(1+\beta)\rho\boldsymbol{e}_\theta \end{cases} \tag{23}$$

$$V_Q = R^{1+2\beta}v \tag{24}$$

Eqs. (9) and (21)–(24) give

$$\begin{cases} \boldsymbol{T}^R = 2nA^nR^{1-(2n-1)(\alpha-\beta)+\alpha}u^{n-1}Y(\rho\omega'\boldsymbol{e}_r - \rho'\boldsymbol{e}_\theta) \\ \boldsymbol{T}^\xi = 2nA^nR^{-(2n-1)(\alpha-\beta)}u^{n-1}[\rho'\boldsymbol{e}_r + \rho\omega'\boldsymbol{e}_\theta + R^\alpha(1+\beta)\rho Y\boldsymbol{e}_\theta] \end{cases} \tag{25}$$

Noting that

$$\begin{cases} \frac{\partial \boldsymbol{e}_r}{\partial R} = \frac{\partial \boldsymbol{e}_\theta}{\partial R} = 0 \\ \frac{\partial \boldsymbol{e}_r}{\partial \xi} = \omega' \boldsymbol{e}_\theta, \qquad \frac{\partial \boldsymbol{e}_\theta}{\partial \xi} = -\omega' \boldsymbol{e}_r \end{cases} \tag{26}$$

Substituting Eqs. (25), (26) into Eq. (11), it follows

$$\begin{cases} (n-1)\frac{u'}{u}\rho' + \rho'' - \rho\omega'^2 - R^\alpha 2(n-1)(\alpha-\beta)Y\rho\omega' = 0 \\ (n-1)\frac{u'}{u}\rho\omega' + \rho\omega'' + 2\rho'\omega' + R^\alpha 2(n-1)(\alpha-\beta)Y\rho' + R^\alpha(1+\beta)\rho[(n-1)\frac{u'}{u}Y - \frac{C}{A}(uv^{-3})'] = 0 \end{cases} \tag{27}$$

Eq. (27) can give

$$\left(n-\frac{1}{2}\right)u' + R^\alpha v\left[(n-1)\frac{u'}{u}Y - \frac{C}{A}(uv^{-3})'\right] = 0 \tag{28}$$

Only taking the dominant terms, Eq. (28) can be reduced to

$$\frac{u'}{u} = -\frac{6R^\alpha}{2n-1}\frac{C}{A}\frac{v'}{v^3} = \frac{6R^\alpha}{2n-1}\frac{C}{A}v^{-2}\left(\frac{\omega''}{\omega'} + \frac{2\rho'}{\rho}\right) \tag{29}$$

Eq. (29) indicates that $u' \sim R^\alpha$, then the second of Eq. (27) gives

$$\rho\omega'' + 2\rho'\omega' \sim R^\alpha \tag{30}$$

Eqs. (29) and (30) are combined to give

$$u' \sim R^{2\alpha} \tag{31}$$

then Eq. (27) is reduced to

$$\begin{cases} \rho'' - \rho\omega'^2 - R^\alpha 2(n-1)(\alpha-\beta)\rho\omega' Y = 0 \\ \rho\omega'' + 2\rho'\omega' + R^\alpha 2(n-1)(\alpha-\beta)\rho' Y = 0 \end{cases} \tag{32}$$

When the terms with R^α are neglected, Eq. (32) becomes

$$\begin{cases} \rho'' - \rho\omega'^2 = 0 \\ \omega'' + 2\frac{\rho'}{\rho}\omega' = 0 \end{cases} \tag{33}$$

The solution to Eq. (33) is

$$\begin{cases} \rho = \rho_0[1 + k^2(\xi+\xi_0)^2]^{1/2} \\ \omega = \omega_0 + \arctan[k(\xi+\xi_0)] \end{cases} \tag{34}$$

in which ρ_0, ω_0, ξ_0 and k are constants. The general form of solution (34) is valid for both material 1 ($\Theta > 0$) and material 2 ($\Theta > 0$).

It can be seen latter that the approximate Eq. (33) is acceptable except when $\xi \to \pm\infty$.

5. Connecting conditions on interface

Shown in Fig. 2 is the crack tip domain. In this paper only the case of $n_1 = n_2$ is considered. It is assumed that the interface is located in the expanding sector.

The solution (34) for material 1 and 2 are written as

$$\begin{cases} \rho_1 = \rho_{10}[1 + k_1^2(\xi+\xi_{10})^2]^{1/2} \\ \omega_1 = \omega_{10} + \arctan[k_1(\xi+\xi_{10})], \quad \xi > 0 \end{cases} \tag{35}$$

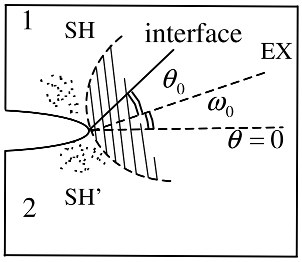

Fig. 2. The crack tip domain.

$$\begin{cases} \rho_2 = \rho_{20}[1 + k_2^2(\xi + \xi_{20})^2]^{1/2} \\ \omega_2 = \omega_{20} + \arctan[k_2(\xi + \xi_{20})], \quad \xi < 0 \end{cases} \tag{36}$$

at $\xi = 0$, the continuity conditions for displacement and stress must be met

$$\begin{cases} \rho_1(0) = \rho_2(0), \qquad \omega_1(0) = \omega_2(0) \\ \boldsymbol{T}_1^{\xi}(0) = \boldsymbol{T}_2^{\xi}(0) \end{cases} \tag{37}$$

From Eqs. (35)–(37) and (25), for the dominant terms, it follows

$$\rho_{10}[1 + k_1^2\xi_{10}^2]^{1/2} = \rho_{20}[1 + k_2^2\xi_{20}^2]^{1/2} \tag{38}$$

$$\omega_{10} + \arctan(k_1\xi_{10}) = \omega_{20} + \arctan(k_2\xi_{20}) \tag{39}$$

$$A_1^n(u_1^{n-1}\rho_1')_{\xi=0} = A_2^n(u_2^{n-1}\rho_2')_{\xi=0} \tag{40}$$

$$A_1^n(u_1^{n-1}\rho_1\omega_1')_{\xi=0} = A_2^n(u_2^{n-1}\rho_2\omega_2')_{\xi=0} \tag{41}$$

Eqs. (40) and (41) give

$$\left(\frac{\rho_1'}{\omega_1'}\right)_{\xi=0} = \left(\frac{\rho_2'}{\omega_2'}\right)_{\xi=0} \tag{42}$$

Eqs. (35), (36), (42) and the first of Eq. (37) give

$$k_1\xi_{10} = k_2\xi_{20} = h \tag{43}$$

where h is a free constant.

Eqs. (35), (36), (43) and the first two of Eq. (37) give

$$\rho_{10} = \rho_{20} = \rho_0, \qquad \omega_{10} = \omega_{20} = \omega_0 \tag{44}$$

ρ_0 and ω_0 are free constants.

Nothing that

$$\begin{cases} u = k^2\rho_0^2, \qquad v = (1+\beta)k\rho_0^2 \\ \omega' = \rho_0^2 k\rho^{-2}, \qquad \rho' = \rho_0^2 k^2(\xi + \xi_0)\rho^{-1} \end{cases} \tag{45}$$

substituting Eqs. (43)–(45) into Eq. (41), it follows

$$k_1 = kA_2^{n/(2n-1)} = ka_1, \qquad k_2 = kA_1^{n/(2n-1)} = ka_2 \tag{46}$$

in which k is a free constant.

Using Eqs. (43), (44) and (46), Eqs. (35) and (36) can be rewritten as

$$\begin{cases} \rho_1 = \rho_0[1 + (a_1 k\xi + h)^2]^{1/2} \\ \omega_1 = \omega_0 + \arctan(a_1 k\xi + h) \end{cases} \tag{47}$$

$$\begin{cases} \rho_2 = \rho_0[1 + (a_2 k\xi + h)^2]^{1/2} \\ \omega_2 = \omega_0 + \arctan(a_2 k\xi + h) \end{cases} \tag{48}$$

Eqs. (47) and (48) meet all of the continuity conditions on the interface, and there are four free constants, ρ_0, ω_0, k, h. The physical meaning of the constants should be discussed. Evidently ρ_0 can indicate the amplitude of the field. When $\xi \to \pm\infty$, $\omega \to \ \omega_0 \pm \pi/2$, so ω_0 indicate the location of the normal line of the crack surface just at the tip, as shown in Fig. 2. Further let

$$h = \tan\theta_0 \tag{49}$$

then at $\xi = 0$

$$\omega_1 = \omega_2 = \omega_0 + \theta_0 \tag{50}$$

Eq. (50) shows that θ_0 is the angle from the normal line (at crack tip) to the interface. We can presume that θ_0 will depend on the ratio of tensile and shear loads. Although the load ratio cannot be determined by the asymptotic solution nevertheless the ratio of tensile and shear stresses on the interface in the near field can be related with θ_0. From the second of Eq. (25) we found that

$$\left.\frac{\sigma^{\xi r}}{\sigma^{\xi\theta}}\right|_{\xi=0} = \left.\frac{\rho'}{\rho\omega'}\right|_{\xi=0} = k\xi_0 = h = \tan\theta_0 \tag{51}$$

so, indeed θ_0 indicates the load ratio.

Finally we consider k $(= k_1/a_1 = k_2/a_2)$. Eq. (41) only ensures the continuity of $\sigma^{\xi\theta}$ for the dominant term, according to Eq. (25), when the term with R^α is considered, the continuity condition of $\sigma^{\xi\theta}$ requires

$$[A_1^n u_1^{n-1} Y_1]_{\xi=0} = [A_2^n u_2^{n-1} Y_2]_{\xi=0} \tag{52}$$

Eqs. (22), (45), (46) and (52) give

$$\frac{1}{a_1}\left[\frac{B_1}{2A_1} + \frac{C_1}{a_1 A_1}\frac{k^{-1}}{(1+\beta)^3}\rho_0^{-4}\right] = \frac{1}{a_2}\left[\frac{B_2}{2A_2} + \frac{C_2}{a_2 A_2}\frac{k^{-1}}{(1+\beta)^3}\rho_0^{-4}\right] \tag{53}$$

It seems that k can be determined by Eq. (53) when $A_1 C_1^{2n-1} \neq A_2 C_2^{2n-1}$. On the other hand, there is no reason to give this restriction on material constants. Therefore the constant k remains an open question.

6. Shrinking sector

The solution (35) and (36) when $\xi \to \pm\infty$ become invalid because $\rho \to \infty$. Then the mapping function should be considered in the shrinking sectors SH or SH′. For simplicity only sector SH is analysed. The mapping functions are assumed as

$$r = R^{1-\delta}\varphi(\Theta), \qquad \theta = \theta^* - R^\gamma \psi(\Theta) \tag{54}$$

where δ and γ are positive exponents, θ^* is a constant.

The physical meaning of Eq. (54) is that the angle Θ is shrinked tremendously while the length of R is extended tremendously.

According to Eqs. (1) and (54) it follows

$$\begin{cases} \boldsymbol{Q}_R = \frac{\partial \boldsymbol{Q}}{\partial R} = R^{-\delta}\varphi[(1-\delta)\boldsymbol{e}_r - \gamma R^{\gamma}\psi \boldsymbol{e}_\theta] \\ \boldsymbol{Q}_\Theta = \frac{\partial \boldsymbol{Q}}{\partial \Theta} = R^{1-\delta}(\varphi' \boldsymbol{e}_r - R^{\gamma}\varphi\psi' \boldsymbol{e}_\theta) \end{cases} \tag{55}$$

then

$$\begin{cases} \boldsymbol{Q}^R = R^{\delta-\gamma}q^{-1}(-\varphi' \boldsymbol{e}_r - R^{\gamma}\varphi\psi' \boldsymbol{e}_\theta) \\ \boldsymbol{Q}^\Theta = R^{\delta-\gamma-1}q^{-1}\varphi[(1-\delta)\boldsymbol{e}_r + \gamma R^{\gamma}\psi \boldsymbol{e}_\theta] \end{cases} \tag{56}$$

$$\text{where } q = \varphi[\gamma\varphi'\psi - (1-\delta)\varphi\psi'] \tag{57}$$

According to Eqs. (2) and (55), and nothing that $\boldsymbol{P}_R = \boldsymbol{e}_R, \boldsymbol{P}_\Theta = R\ \boldsymbol{e}_\Theta$, it follows

$$I = R^{2\delta}p, \qquad J = R^{\gamma-2\delta}q \tag{58}$$

in which

$$p = \varphi'^2 + (1-\delta)^2\varphi^2 \tag{59}$$

Besides, from Eqs. (10) and (55)

$$V_Q = R^{1+\gamma-2\delta}q \tag{60}$$

For the time being it is assumed that $I \gg J$, $J \gg 1$, then expression (19) is still valid. From Eqs. (9), (19) and (6) it follows that

$$\begin{cases} \boldsymbol{T}^R = 2nA^nJ^{-1}V_QI^{n-1}[\boldsymbol{Q}_R + J(\frac{B}{2A} - \frac{C}{A}IJ^{-3})\boldsymbol{Q}^R] \\ \boldsymbol{T}^\Theta = 2nA^nJ^{-1}V_QI^{n-1}[R^{-2}\boldsymbol{Q}_\Theta + J(\frac{B}{2A} - \frac{C}{A}IJ^{-3})\boldsymbol{Q}^\Theta] \end{cases} \tag{61}$$

Eqs. (58)–(61) can give

$$\begin{cases} \boldsymbol{T}_R = 2nA^n \quad R^{1-(2n-1)\delta}p^{n-1}\{\varphi[(1-\delta)\boldsymbol{e}_r - \gamma R^{\gamma}\psi \boldsymbol{e}_\theta] - (\frac{B}{2A} - \frac{C}{A}pq^{-3}R^{4\delta-3\gamma})(\varphi' \boldsymbol{e}_\theta + R^{\gamma}\varphi\psi' \boldsymbol{e}_r)\} \\ \boldsymbol{T}^\Theta = 2nA^n \quad R^{-(2n-1)\delta}p^{n-1}\{\varphi' \boldsymbol{e}_r - \gamma R^{\gamma}\varphi\psi' \boldsymbol{e}_\theta + (\frac{B}{2A} - \frac{C}{A}pq^{-3}R^{4\delta-3\gamma})\varphi[(1-\delta)\boldsymbol{e}_\theta + \gamma R^{\gamma}\psi \boldsymbol{e}_r]\} \end{cases} \tag{62}$$

In order to simplify Eq. (62), the boundary condition at the crack surface $\Theta = \pi$ should be considered. The traction free condition is

$$\boldsymbol{T}^\Theta|_{\Theta=\pi} = 0 \tag{63}$$

Eqs. (62) and (63) are combined to give

$$\varphi'(\pi) = 0 \tag{64}$$

$$\left(\frac{B}{2} - Cpq^{-3}R^{4\delta-3\gamma}\right)_{\Theta=\pi} = 0 \tag{65}$$

Eq. (65) requires that

$$\gamma = \frac{4}{3}\delta \tag{66}$$

$$q = \left(\frac{2C}{B}p\right)^{1/3} \quad \text{at } \Theta = \pi \tag{67}$$

Let

$$G = \frac{B}{2A} - \frac{C}{A} p q^{-3} \tag{68}$$

then Eq. (62) is reduced to

$$\begin{cases} \boldsymbol{T}^R = 2nA^n R^{1-(2n-1)\delta} p^{n-1}[(1-\delta)\varphi \boldsymbol{e}_r - \varphi' G \boldsymbol{e}_\theta] \\ \boldsymbol{T}^\Theta = 2nA^n R^{-(2n-1)\delta} p^{n-1}[\varphi' \boldsymbol{e}_r + (1-\delta)\varphi G \boldsymbol{e}_\theta] \end{cases} \tag{69}$$

Substituting Eq. (69) into Eq. (11), it follows

$$\varphi'' + (n-1)\varphi' \frac{p'}{p} + (1-\delta)[1-(2n-1)\delta]\varphi = 0 \tag{70}$$

$$(1-\delta)\varphi G' + (1-\delta)(n-1)\varphi G \frac{p'}{p} + 2(n-1)\delta \varphi' G = 0 \tag{71}$$

Eqs. (70) and (71) can be reduced to

$$[\varphi'' + (1-\delta)^2 \varphi][1 + 2(n-1)\varphi'^2 p^{-1}] - 2(n-1)\delta(1-\delta)\varphi = 0 \tag{72}$$

$$\frac{G'}{G} + (n-1)\frac{p'}{p} + \frac{2\delta}{1-\delta}(n-1)\frac{\varphi'}{\varphi} = 0 \tag{73}$$

The solution of Eq. (73) is

$$G = G_0 (p \varphi^{2\delta/(1-\delta)})^{n-1} \tag{74}$$

where G_0 is a constant. Since $p, \varphi > 0$, Eq. (68) and the boundary condition (65) require $G_0 = 0$, then $G \equiv 0$, and Eq. (68) gives

$$q = \left(\frac{2C}{B} p \right)^{1/3} \tag{75}$$

Besides Eqs. (64), in order to connect with the EX sector, the following condition is needed

$$\varphi(0) = 0 \tag{76}$$

The nonlinear eigenvalue problem of Eq. (72) under conditions (64) and (76) can be solved analytically (Gao and Gao, 1999)

$$\begin{cases} \varphi = \varphi^* (\Omega - \cos\Theta)^{1/2} [\Omega + (1 - \frac{1}{n})\cos\Theta]^{1/2-\delta} \\ \Omega = [1 - (1 - \frac{1}{n})^2 \sin^2\Theta]^{1/2} \\ \delta = \frac{1}{2n} \end{cases} \tag{77}$$

where φ^* is a constant.

The solution of Eq. (75) is

$$\psi = \frac{1}{1-\delta} \left(\frac{2C}{B} \right)^{1/3} \varphi^{\gamma/(1-\delta)} \left[\int_\Theta^\pi \varphi^{-2\gamma/(1-\delta)} p^{1/3} \mathrm{d}x + \psi^* \right] \tag{78}$$

where ψ^* is a free parameter.

The solution for the material 2 ($\Theta < 0$) can be similarly obtained.

7. Transition from sector EX to SH or SH′

The completely analytical solutions are obtained for sector EX and SH or SH′ respectively. However, there are no strict boundaries between EX and SH or SH′. Therefore the transition from one sector to another must be discussed. For briefness, only the transition between EX and SH is considered in details.

Firstly, we try to use the simple solution (34) to express the asymptotic behavior of ρ and ω when $\xi \to +\infty$, then

$$\begin{cases} \rho = \rho_0 k \xi \\ \omega = \omega_0 + \frac{\pi}{2} - \frac{1}{k\xi}, \quad \xi \to +\infty \end{cases} \tag{79}$$

Substituting Eq. (79) into Eq. (12), it follows

$$\begin{cases} r = R^{1+\beta-\alpha} \rho_0 k \Theta \\ \theta = \omega_0 + \frac{\pi}{2} - R^{\alpha} \frac{1}{k\Theta}, \quad \Theta \to 0 \end{cases} \tag{80}$$

On the other hand, for sector SH, when $\Theta \to 0$, Eq. (77) gives

$$\varphi = C_\varphi \Theta \tag{81}$$

where

$$C_\varphi = \varphi^* \delta^{1/2} [2(1-\delta)]^{1-\delta} = \varphi(\pi) \delta^{\delta} (1-\delta)^{1-\delta} \tag{82}$$

Eqs. (81) and (78) give

$$\psi = C_\psi \Theta^{-1} \tag{83}$$

where

$$C_\psi = \left(\frac{2C}{B}\right)^{1/3} \left(1 + \frac{\delta}{3}\right)^{-1} C_\varphi^{-4/3} \tag{84}$$

Substituting Eqs. (81) and (83) into Eq. (54), it follows

$$\begin{cases} r = R^{1-\delta} C_\varphi \Theta \\ \theta = \theta^* - R^{\gamma} C_\psi \Theta^{-1}, \quad \Theta \to 0 \end{cases} \tag{85}$$

Eqs. (80) and (85) should give the some mapping from R, Θ to r, θ, therefore it is required

$$\alpha = \gamma, \qquad \beta = \gamma - \delta \tag{86}$$

$$\omega_0 = \theta^* - \frac{\pi}{2} \tag{87}$$

$$k = \frac{1}{C_\psi}, \qquad \rho_0 = C_\psi \cdot C_\varphi \tag{88}$$

Eqs. (86), (66) and (20) are consistent.
Eqs. (88) and (84) are combined to get

$$k \rho_0^4 \left(1 + \frac{\delta}{3}\right)^3 = \frac{2C}{B} \tag{89}$$

Particularly for material 1, Eq. (89) is written as

$$k_1\rho_0^4\left(1+\frac{\delta}{3}\right)^3=\frac{2C_1}{B_1} \quad (90)$$

Similarly, for material 2, it is

$$k_2\rho_0^4\left(1+\frac{\delta}{3}\right)^3=\frac{2C_2}{B_2} \quad (91)$$

Eqs. (90) and (91) can give

$$\frac{k_1}{k_2}=\frac{C_1B_2}{C_2B_1} \quad (92)$$

Noting Eqs. (46) and (92), it follows

$$\left(\frac{A_2}{A_1}\right)^{n/(2n-1)}=\frac{C_1B_2}{C_2B_1} \quad (93)$$

this is the same condition forced by Herrmann (1989) but from different point of view.

However, Eq. (93) should not be a precondition for existence of the solution. The source of the contradiction will be discussed in next paragraph.

8. The role of terms with R^α in Eq. (32)

Now, we consider the simplifying procedure from Eqs. (32) and (33). Under consideration is the case of $R \ll 1$, so it is reasonable to neglect the terms with R^α in Eq. (32). However, at some particular points, for example $\xi \to \pm\infty$, the terms with R^α may play an important role. In order to show it clearly, the new variable ζ is introduced

$$\zeta = \arctan\xi \quad (94)$$

then Eq. (32) is written as

$$\begin{cases} \ddot{\rho}-2\tan\zeta\dot{\rho}-\rho\dot{\omega}^2-\frac{R^\alpha}{\cos^2\zeta}(1-\frac{1}{n})\rho\dot{\omega}Y=0 \\ \ddot{\omega}-2\tan\zeta\dot{\omega}+2\frac{\dot{\rho}}{\rho}\dot{\omega}+\frac{R^\alpha}{\cos^2\zeta}(1-\frac{1}{n})\frac{\dot{\rho}}{\rho}Y=0 \end{cases} \quad (95)$$

in which

$$(\dot{\ }) = \frac{\mathrm{d}}{\mathrm{d}\zeta}(\ ) \quad (96)$$

$$Y=\frac{B}{2A}-\frac{C}{A}\hat{u}\hat{v}^{-3}\cos^{-2}\zeta \quad (97)$$

$$\hat{u}=\dot{\rho}^2+\rho\dot{\omega}^2 \quad (98)$$

$$\hat{v}=(1+\beta)\rho^2\dot{\omega} \quad (99)$$

From Eq. (95) it can be seen that when $\zeta \to \pm\pi/2(\xi \to \pm\infty)$ the tails of Eq. (95) with R^α will become important. Therefore Eq. (33) can only be used for the interval $|\zeta| < (\pi/2) - \Delta\zeta$, the value of $\Delta\zeta$ depends on R^α. When $\zeta \to \pm\pi/2(\xi \to \pm\infty)$, the solution (34) cannot describe the real behavior of ρ and $\dot{\omega}$. Analysis of Eq. (95) concludes that for a given value of R^α, when $\zeta \to \pm\pi/2$, the only possiblity is

$$\begin{cases} \rho = C_\rho(\frac{\pi}{2} - |\zeta|)^{-1} \\ \dot{\omega} = C_\omega \end{cases} \tag{100}$$

and

$$Y \to 0 \tag{101}$$

Further, Eqs. (98) and (99) become

$$\begin{cases} \hat{u} = C_\rho^2(\frac{\pi}{2} - |\zeta|)^{-4} \\ \hat{v} = (1+\beta)C_\rho^2 C_\omega(\frac{\pi}{2} - |\zeta|)^{-2} \end{cases} \tag{102}$$

Eqs. (97), (101) and (102) give

$$C_\omega^3 C_\rho^4 = \frac{2C}{B(1+\beta)^3} \tag{103}$$

The asymptotic behavior of Eq. (101) is independent of the initial value of ρ and ω' at $\zeta = 0$. Eq. (101) is also verified by numerical calculation of Eq. (95) for different value of R^α. The curves of Y are calculated for $n = 2, A = B = C, \rho(0) = 1$ and various R^α. For $\omega'(0) = 1$ and $\omega'(0) = 2$ the curves are plotted in Figs. 3, 4 respectively. Finally, using Eq. (94), Eq. (100) can be written as

$$\begin{cases} \rho = C_\rho|\xi| \\ \omega' = C_\omega \xi^{-2}, \quad \xi \to \pm\infty \end{cases} \tag{104}$$

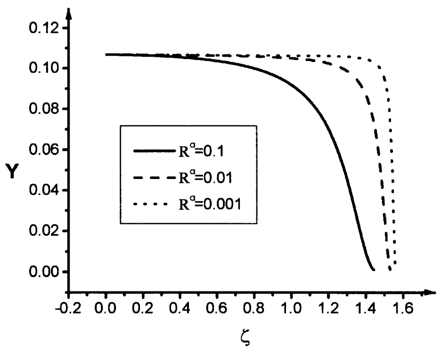

Fig. 3. The curves of Y for $\omega'(0) = 1$.

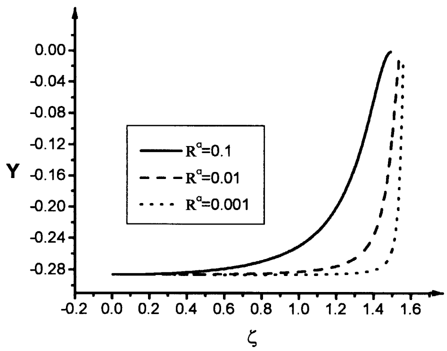

Fig. 4. The curves of Y for $\omega'(0) = 2$.

The asymptotic expression (104) is similar to Eq. (79), but generally $C_\rho \neq \rho_0 k, C_\omega \neq k^{-1}$. This transition mainly happens in the zone $|\zeta| \sim \pi/2 - R^{\alpha/2}$. From Eqs. (104) and (12), it follows

$$\begin{cases} r = R^{1+\beta-\alpha} C_\rho |\Theta| \\ \theta = \omega_0 \pm \frac{\pi}{2} \mp R^\alpha C_\omega \Theta^{-1}, \quad \Theta \to 0 \end{cases} \tag{105}$$

Comparing Eqs. (85) and (105), we can obtain Eqs. (86) and (87) but Eq. (88) is replaced by

$$C_\phi = C_\rho, \qquad C_\psi = C_\omega \tag{106}$$

Although C_ϕ and C_ψ are related by Eq. (84), nevertheless when C_ρ and C_ω are given, Eq. (106) can be satisfied, because C_ρ and C_ω are similarly related by Eq. (103). Therefore through the rapid transition zone, the asymptotic behavior of Eq. (80) is replaced by Eq. (105), then condition (93) is released. With the help of the minor terms of Eq. (32), the obstacle of linking different sectors is overcome.

It should be noted that for a crack in uniform material, the value of k can be taken such that $Y(0) = 0$, then the rapid transition zone disappeared. Only for the interface crack the rapid transition zone is needed.

It should be noted that when $(\pi/2) - |\zeta| \gg R^{\alpha/2}$, the solution (47) and (48) are valid, so the stress state is uniaxial tension.

9. Comparison with results of Gao and Shi (1995)

Analysed in Gao and Shi (1995) is also an interface crack tip, but the elastic law was proposed by Gao (1990).

$$U = a(IJ^{-2/3})^N + b(J^2 - 1)^m J^{-2t} \tag{107}$$

In sector SH or SH′ it was obtained that

$$q = \left(\frac{Na}{3sb} p^N\right)^{3/2(3s+N)}, \quad s = m - t \tag{108}$$

Eq. (108) is similar to relation (75).

In sector EX, the equation of Gao and Shi (1995) is

$$\begin{cases} \rho'' - \rho\omega'^2 - f(u, v, \rho'^2) Y^* = 0 \\ \omega'' + 2\frac{\rho'}{\rho}\omega' - g(u, v, \rho'^2) Y^* = 0 \end{cases} \tag{109}$$

in which

$$Y^* = 1 - \frac{3sb}{Na} v^{2s+(2/3)N} u^{-N} \tag{110}$$

f and g are functions u, v, ρ'^2. The terms with Y^* in Eq. (109) are similar to the tails in Eq. (32) but without the factor R^α. Therefore they were not neglected in the analysis of Gao and Shi (1995).

For a crack in uniform material, Y^* can simply be taken to be zero, then

$$v = \left(\frac{Na}{3sb} u^N\right)^{3/2(3s+N)} \tag{111}$$

Therefore the tails of Eq. (109) are cut off, and the analytical solution can be given by Eq. (34). The condition (111) can be satisfied if

$$k = \left[(1+\beta)^{-(3s+N)} \left(\frac{Na}{3sb} \right)^{3/2} \rho_0^{-(6s-N)} \right]^{1/(3s-2N)} \tag{112}$$

Eqs. (108) and (111) can ensure the reasonable connection of sector EX with SH or SH′. Therefore, for the crack in uniform material, the crack tip is in uniaxial tension state.

For an interface crack, the tails of Eq. (109) cannot be simply cut off. With the help of the tails, when $\xi \to \pm\infty$, Y^* will tend to zero automatically so that Eq. (111) is satisfied. So, the role of the terms with Y^* in Eq. (109) is similar with Y in Eq. (32). For an interface crack, generally, the stress state at crack tip is not in uniaxial tension.

10. Conclusions

(1) The interface crack tip field, for K–S elastic law when $n_1 = n_2$, is composed of two shrinking sectors and one expanding sector. The interface is located in expanding sector.

(2) For shrinking sectors the analytical solution is obtained both for φ and ψ. For expanding sector in the main domain, i.e. $|\xi| \ll R^{-\alpha/2}$, the analytical solution is valid. But there is a rapid transition zone at $|\xi| \sim R^{-\alpha/2}$, where the minor terms R^α in Eq. (32) must be considered. With the help of the tails of Eq. (32), the transition from sector SH to EX and from EX to SH′ can be performed, i.e. the displacements and stress are continuous.

(3) Although the minor terms with R^α play an important role, the stress state near the interface crack tip is still in uniaxial tension. This result is different from that obtained by Gao and Shi (1995) for another elastic law.

(4) In the works of Herrmann (1989, 1992) the important expanding sector was ignored so that some very long analysis are caused. Besides, the restriction on the material constants is not necessary.

(5) For the interface crack tip field under mixed load, the asymptotic solution mainly contains three free parameters. ρ_0 indicate the amplitude of the field. ω_0 represents the orientation of the field. θ_0 is related with load ratio. Although Eq. (53) provides a condition to determine the value of k, but this is uncertain. So, the determination of constant k may need an additional condition that cannot be given by the asymptotic solution, see Gao and Gao (1999).

(6) When large strain is considered, the oscillatory singularity disappeared.

(7) The approach used in this paper is limited in the case of $n_1 = n_2 = n$.

Acknowledgements

This work is supported by the National Science Foundation of China (19572001).

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